Generalized Schwarzian derivatives and higher order differential equations

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Abstract

It is shown that the well-known connection between the second order linear differential equation $h''+B(z)\,h=0$, with a solution base $\{h_1,h_2\}$, and the Schwarzian derivative

$$S_f=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

of $f=h_1/h_2$, can be extended to the equation $h^{(k)}+B(z)\,h=0$ where $k\geq 2$. This generalization depends upon an appropriate definition of the generalized Schwarzian derivative $S_k(f)$ of a function f which is induced by $k-1$ ratios of linearly independent solutions of $h^{(k)}+B(z)\,h=0$. The class $\mathcal{R}_k(\Omega)$ of meromorphic functions f such that $S_k(f)$ is analytic in a given domain Ω is also completely described. It is shown that if Ω is the unit disc $\mathbb{D}$ or the complex plane $\mathbb{C}$, then the order of growth of $f\in\mathcal{R}_k(\Omega)$ is precisely determined by the growth of $S_k(f)$, and vice versa. Also the oscillation of solutions of $h^{(k)}+B(z)\,h=0$, with the analytic coefficient B in $\mathbb{D}$ or $\mathbb{C}$, in terms of the exponent of convergence of solutions is briefly discussed.

1. Introduction and results

Let $\mathbb{D}$ denote the unit disc of the complex plane $\mathbb{C}$, and let $\mathcal{M}(\Omega)$ and $\mathcal{H}(\Omega)$ stand for the sets of all meromorphic and analytic functions in a domain $\Omega\subset\mathbb{C}$, respectively. If there is no need to specify the domain, we will simply write $f\in\mathcal{M}$ or $f\in\mathcal{H}$.

We say that $f\in\mathcal{M}(\Omega)$ belongs to the *restricted class* $\mathcal{R}(\Omega)$, if f has only simple poles and $f'(z)\neq 0$ for all $z\in\Omega$. As in the case of $\mathcal{M}$, we will write $f\in\mathcal{R}$ if the domain Ω does not have to be specified. The *Schwarzian derivative* of $f\in\mathcal{R}$ at z is defined as

$$S_f(z):=\left(\frac{f''}{f'}\right)'(z)-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

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The Schwarzian derivative S_f measures how much f differs from being a Möbius transformation. In particular, $S_f \equiv 0$ if and only if f is a Möbius transformation. It is also clear that $S_f \in \mathcal{H}$ if $f \in \mathcal{R}$. Moreover, if $f \in \mathcal{M}(\Omega)$ and $h \in \mathcal{H}(\Omega)$ is locally univalent such that $h(\Omega) \subset \Omega$, then

$$S_{f\circ h}(z) = S_f(h(z))(h'(z))^2 + S_h(z) \tag{1.1}$$

for all $z \in \Omega$.

An important property of the Schwarzian derivative is its well-known connection to second order linear differential equations.

Theorem A. *Let $B \in \mathcal{H}$. Then the quotient $f := h_1/h_2$ of any linearly independent solutions h_1 and h_2 of*

$$h'' + B(z)\,h = 0 \tag{1.2}$$

belongs to $\mathcal{R}$, and $S_f = 2B$.

Conversely, let $f \in \mathcal{R}$ and define $B := \frac{1}{2}S_2(f)$. Then $B \in \mathcal{H}$ and (1.2) admits linearly independent solutions h_1 and h_2 such that $f = h_1/h_2$.

Generalized Schwarzian derivatives

Let $f \in \mathcal{M}$ and consider the meromorphic functions defined by the formulas

$$S_{2,n}(f) := \frac{f''}{f'}, \qquad S_{k+1,n}(f) := \big(S_{k,n}(f)\big)' - \frac{1}{n}\frac{f''}{f'}S_{k,n}(f), \quad n \in \mathbb{N},\, k \in \mathbb{N}\setminus\{1\}\,,$$

and

$$S_k(f) := S_{k+1,k}(f), \quad k \in \mathbb{N}.$$

Note that the definition of $S_{2,n}(f)$ is independent of n. Then $S_1(f)$ is the *pre-Schwarzian derivative* of f, and

$$S_2(f) = S_{3,2}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = S_f.$$

Therefore $S_k(f)$ can be called a *generalized Schwarzian derivative* of f.

Direct calculations show that

$$S_3(f) = \frac{f^{(4)}}{f'} - 4\left(\frac{f'''}{f'}\right)\left(\frac{f''}{f'}\right) + \frac{28}{9}\left(\frac{f''}{f'}\right)^3$$

and

$$S_4(f) = \frac{f^{(5)}}{f'} - 5\left(\frac{f^{(4)}}{f'}\right)\left(\frac{f''}{f'}\right) + \frac{135}{8}\left(\frac{f^{(3)}}{f'}\right)\left(\frac{f''}{f'}\right)^2 - \frac{15}{4}\left(\frac{f^{(3)}}{f'}\right)^2 - \frac{585}{64}\left(\frac{f''}{f'}\right)^4.$$

In each term of $S_3(f)$ (resp. $S_4(f)$) the sum of the differences between the orders of the derivatives in the numerator and the denominator is exactly 3 (resp. 4). Other Schwarzian derivatives also share this property in the sense that the corresponding sum in the case of $S_k(f)$ is always k.

One can find different definitions of higher order Schwarzian derivatives in the existing literature. In particular, $\sigma_{k+1}(f)$, defined in [19], is closely related to $S_k(f)$. One can show that each term in $\sigma_{k+1}(f)$ is a constant multiple of the corresponding term in $S_k(f)$, yet obviously $\sigma_{k+1}(f) \neq S_k(f)$ unless $k = 2$. The Schwarzians $\sigma_{k+1}(f)$ have nice properties with regards to compositions of functions whereas the functions $S_k(f)$ do not. Especially, a formula similar to (1.1) can be established for $\sigma_{k+1}(f)$, see [19, p. 3242]. Another definition of a generalized Schwarzian derivative can be found in [3]. The definition given in the present paper appears to give a natural connection to higher order linear differential equations in the spirit of Theorem A.

Definition 1. Let $f \in \mathcal{M}$ and $k \in \mathbb{N}$. Then f belongs to the *k-restricted class* $\mathcal{R}_k$, if f' can be represented in the form $f' = 1/h^k$, where $h \in \mathcal{H}$ admits the following properties:

(i) zeros of h are at most $(k-1)$-fold;

(ii) at each l-fold zero of h all derivatives $h^{(k)}, \dots, h^{(k+l-1)}$ vanish.

Condition (ii) in Definition 1 says that if h has an l-fold zero at α, then $h^{(k)}$ has to have at least an l-fold zero at α. This kind of functions appear naturally in the theory of differential equations.

Example 1. Every solution h of

$$h^{(k)} + B(z)\, h = 0, \tag{1.3}$$

where $B \in \mathcal{H}$, satisfies properties (i) and (ii) in Definition 1. To prove (i), assume on the contrary that h has an m-fold zero at α, and $m \geq k$. Then, in a neighborhood of α, $h(z) = (z-\alpha)^m\, H(z)$, where $H \in \mathcal{H}$ and $H(\alpha) \neq 0$. Therefore $h^{(k)}(z) = (z-\alpha)^{m-k}\, K(z)$, where $K \in \mathcal{H}$ and $K(\alpha) \neq 0$. As h is a solution of (1.3),

$$B(z) = -\frac{h^{(k)}(z)}{h(z)} = \frac{1}{(z-\alpha)^k}\,\frac{K(z)}{H(z)},$$

where K/H is analytic in a neighborhood of α and $K(\alpha)/H(\alpha) \neq 0$. Thus B has a pole of order k at α, which contradicts the assumption $B \in \mathcal{H}$. Property (ii) follows by $l-1$ differentiations of (1.3) because $B \in \mathcal{H}$.

Obviously $\mathcal{R}_1$ is just the class of locally univalent analytic functions. The connection between the restricted class $\mathcal{R}$ and $\mathcal{R}_2$ is given in the following lemma whose proof and other lengthy reasonings are postponed to forthcoming sections.

Lemma 2. *The classes $\mathcal{R}$ and $\mathcal{R}_2$ are equal.*

We next give concrete examples of functions in $\mathcal{R}_3$ and $\mathcal{R}_4$.

Example 2. Consider the meromorphic function

$$f_1(z) = -\frac{1}{5z^5} - \frac{1}{2z^2}, \quad f_1'(z) = \frac{1}{z^6} + \frac{1}{z^3} = \frac{1}{(h_1(z))^3}, \quad h_1(z) = \frac{z^2}{(1+z^3)^{\frac{1}{3}}},$$

where $h_1 \in \mathcal{H}(\mathbb{D})$. The zeros of f_1' are the solutions of $z^3 = -1$, and thus f_1' does not vanish in $\mathbb{D}$. Calculations show that

$$h_1^{(3)}(z) = -\frac{20\,z^2}{(1+z^3)^{\frac{4}{3}}} + \frac{48\,z^5}{(1+z^3)^{\frac{7}{3}}} - \frac{28\,z^8}{(1+z^3)^{\frac{10}{3}}},$$
$$h_1^{(4)}(z) = \frac{320\,z^4}{(1+z^3)^{\frac{7}{3}}} - \frac{40\,z}{(1+z^3)^{\frac{4}{3}}} - \frac{560\,z^7}{(1+z^3)^{\frac{10}{3}}} + \frac{280\,z^{10}}{(1+z^3)^{\frac{13}{3}}},$$

and hence $h_1^{(3)}(0) = h_1^{(4)}(0) = 0$. Therefore $f_1 \in \mathcal{R}_3(\mathbb{D})$ by Definition 1. Further,

$$S_3(f_1)(z) = \frac{60 - 24\,z^3}{(1+z^3)^3},$$

and so $S_3(f_1) \in \mathcal{H}(\mathbb{D})$. One can also show that h_1 is a solution of (1.3) with $k = 3$ and $B = \frac{1}{3}\,S_3(f_1)$.

Consider the meromorphic function

$$f_2(z) = -\frac{1+2\sqrt{2}\,i}{9\,z^3} + \frac{2+\sqrt{2}\,i}{3\,z^2} - \frac{1}{z}, \quad f_2'(z) = \frac{1+2\sqrt{2}\,i}{3\,z^4} - \frac{4+2\sqrt{2}\,i}{3\,z^3} + \frac{1}{z^2},$$

where

$$f_2'(z) = \frac{1}{(h_2(z))^4}, \quad h_2(z) = \frac{3^{\frac{1}{4}}\,z}{((z-1)(3z-1-2\sqrt{2}\,i))^{\frac{1}{4}}}.$$

Now $h_2 \in \mathcal{H}(\mathbb{D})$ and f_2' is non-vanishing in $\mathbb{D}$ since the points 1 and $\frac{1+2\sqrt{2}\,i}{3}$ belong to the boundary of $\mathbb{D}$. Further, a calculation shows that $h_2^{(4)}(0) = 0$, and hence $f_2 \in \mathcal{R}_4(\mathbb{D})$ by Definition 1. Furthermore, one can check that $S_4(f_2) \in \mathcal{H}(\mathbb{D})$ and h_2 is a solution of (1.3) with $k = 4$ and $B = \frac{1}{4}S_4(f_2)$.

The phenomenon related to differential equations which occurs in Example 2 for the functions f_1 and f_2 and their generalized Schwarzian derivatives $S_3(f_1)$ and $S_4(f_2)$ is by no means a casuality. Lemmas 3, 4 and 5 explain the interrelationships between the generalized Schwarzian derivative $S_k(f)$, the k-restricted class $\mathcal{R}_k$, and linear differential equations of order k. This connection is further underscored in Theorem 6, which is the main result of this section.

Lemma 3. *Let $f \in \mathcal{M}$ such that $f' = 1/h^k$ for some $h \in \mathcal{H}$, $h \not\equiv 0$, and $k \in \mathbb{N}$. Then*

$$S_k(f) = -k\,\frac{h^{(k)}}{h}, \tag{1.4}$$

and any constant multiple of $h = (f')^{-1/k}$ is a solution of

$$h^{(k)} + \frac{1}{k}S_k(f)(z)\,h = 0. \tag{1.5}$$

If $f \in \mathcal{R}_k$, then f' is of the form $f' = 1/h^k$, where $h \in \mathcal{H}$. Therefore Lemma 3 connects the generalized Schwarzian derivative $S_k(f)$ to linear differential equations of order k.

If P_{k-1} is a polynomial with $\deg(P_{k-1}) \leq k-1$ and $f' = (P_{k-1})^{-k}$, then $S_k(f) \equiv 0$ by Lemma 3. The converse implication is also true.

Lemma 4. *Let $f \in \mathcal{M}$ such that f' is non-vanishing, and let $k \in \mathbb{N}$. Then $S_k(f) \equiv 0$ if and only if $f' = (P_{k-1})^{-k}$, where P_{k-1} is a polynomial with* $\deg(P_{k-1}) \leq k-1$.

If $k = 1$, then Lemma 4 simply says that the pre-Schwarzian is identically zero if and only if f' is a non-zero constant. The case $k = 2$ reduces to the known fact for the classical Schwarzian derivative since the derivative of a Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \quad \text{is} \quad f'(z) = \left(\frac{c}{\sqrt{ad-bc}}\, z + \frac{d}{\sqrt{ad-bc}} \right)^{-2},$$

where $ad - bc \neq 0$.

The next lemma implies that a function h is a solution of the differential equation (1.3), with some $B \in \mathcal{H}$, if and only if h satisfies the properties (i) and (ii) in Definition 1.

Lemma 5. *Let $f \in \mathcal{M}$ such that f' is non-vanishing. Then the following conditions are equivalent:*

(i) $f \in \mathcal{R}_k$;

(ii) $S_k(f) \in \mathcal{H}$;

(iii) $f' = 1/h^k$, *where h is a solution of* (1.3) *with some* $B \in \mathcal{H}$.

Lemma 5 allows us to describe a natural and large subclass of $\mathcal{R}_k$ in terms of the Laurent series of f'.

Example 3. Let $f \in \mathcal{M}$ and $k \in \mathbb{N}$. We say that $f \in \mathcal{R}_k^\star$, if f' is non-vanishing and it admits the following properties:

(i) poles of f' are of order lk, where $l = 1, \ldots, k-1$;

(ii) if f' has a pole of order lk at α, then its Laurent series in a punctured neighborhood of α is of the form

$$f'(z) = \frac{c_{-lk}}{(z-\alpha)^{lk}} + \sum_{j=-lk+k}^{\infty} c_j (z-\alpha)^j, \quad c_{-lk} \neq 0.$$

The class $\mathcal{R}_k^\star$ is a subset of $\mathcal{R}_k$.

By Example 3 the function f_1 in Example 2 belongs to $\mathcal{R}_3$. Moreover, the function f_2 in Example 2 shows that conditions (i) and (ii) above do not characterize the class $\mathcal{R}_4$.

The following result generalizes Theorem A for higher order equations. If $k = 2$, then the Wronskian determinant $W((h_1/h_2)')$ is $(h_1/h_2)'$, and hence Theorem 6 with $k = 2$ reduces to Theorem A.

Theorem 6. *Let $\{h_1,\dots,h_k\}$ be a solution base of* (1.3) *where $B\in\mathcal{H}$ and $k\geq 2$. Then every primitive function f of the Wronskian determinant*

$$W\left(\left(\frac{h_1}{h_k}\right)',\left(\frac{h_2}{h_k}\right)',\dots,\left(\frac{h_{k-1}}{h_k}\right)'\right) \tag{1.6}$$

belongs to $\mathcal{R}_k$, and $S_k(f)=kB$.

Conversely, let $f\in\mathcal{R}_k$, $k\geq 2$, and define $B:=\frac{1}{k}S_k(f)$. Then $B\in\mathcal{H}$ and (1.3) *admits a solution base $\{h_1,\dots,h_k\}$ such that f is a primitive function of* (1.6).

The first part of Theorem 6 says that the analytic coefficient of (1.3) can be expressed in terms of $k-1$ ratios of linearly independent solutions. An analogous result can be found in the existing literature. Namely, if $\{h_1,\dots,h_k\}$ is a solution base of (1.3), where $B\in\mathcal{H}$, then a special case of [13, Theorem 2.1] yields

$$B=\sum_{j=0}^{k-1}(-1)^{2k-j}\frac{W_{k-j}}{W_k}\frac{\left(\sqrt[k]{W_k}\right)^{(k-j)}}{\sqrt[k]{W_k}},$$

where

$$W_j=\begin{vmatrix} \left(\frac{h_1}{h_k}\right)' & \left(\frac{h_2}{h_k}\right)' & \cdots & \left(\frac{h_{k-1}}{h_k}\right)' \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{h_1}{h_k}\right)^{(j-1)} & \left(\frac{h_2}{h_k}\right)^{(j-1)} & \cdots & \left(\frac{h_{k-1}}{h_k}\right)^{(j-1)} \\ \left(\frac{h_1}{h_k}\right)^{(j+1)} & \left(\frac{h_2}{h_k}\right)^{(j+1)} & \cdots & \left(\frac{h_{k-1}}{h_k}\right)^{(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{h_1}{h_k}\right)^{(k)} & \left(\frac{h_2}{h_k}\right)^{(k)} & \cdots & \left(\frac{h_{k-1}}{h_k}\right)^{(k)} \end{vmatrix}, \quad j=1,\dots,k.$$

This along with the first part of Theorem 6 shows that

$$S_k(f)=k\sum_{j=0}^{k-1}(-1)^{2k-j}\frac{W_{k-j}}{W_k}\frac{\left(\sqrt[k]{W_k}\right)^{(k-j)}}{\sqrt[k]{W_k}},$$

where f is a primitive of the Wronskian (1.6).

Note that the representations of analytic coefficients in terms of $k-1$ ratios of linearly independent solutions given in [13, Theorem 2.1] are valid for equations of the form

$$h^{(k)}+B_{k-2}(z)h^{(k-2)}+\cdots+B_1(z)h'+B_0(z)h=0. \tag{1.7}$$

This suggests that the first part of Theorem 6 should have an analogue for the equation (1.7). However, apart from the fact that the argument used in the proof of Theorem 6 does not seem to work for (1.7), we will face other obstacles. Namely, in view of Lemma 5, it is unclear if each coefficient B_j could be represented in terms of some generalized Schwarzian derivative of some function f (or some generalized Schwarzian derivatives of some functions f_j), induced by ratios of linearly independent solutions, and to which restricted class this f (or these f_j:s) should belong to. It seems that the definition of $S_k(f)$ is not adequate for this purpose unless all the intermediate coefficients vanish identically. Neither it is clear how the second part of Theorem 6 should be stated in the case of (1.7).

Order of growth via generalized Schwarzian derivatives

We next combine the results from the previous section with known results on differential equations to characterize finite order functions in $\mathcal{R}_k(\mathbb{D})$ and $\mathcal{R}_k(\mathbb{C})$ in terms of their generalized Schwarzian derivatives. To do this, several definitions are needed.

The *Nevanlinna characteristic* of $f \in \mathcal{M}(\Omega)$, where Ω is either $\mathbb{D}$ or $\mathbb{C}$, is

$$T(r,f) := m(r,f) + N(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta + \int_0^r \frac{n(t)-n(0)}{t} dt + n(0)\log r,$$

where $m(r,f)$ is the *proximity function* and $N(r,f)$ is the *integrated counting function.* The *orders of growth* of $f \in \mathcal{M}(\mathbb{D})$ and $g \in \mathcal{M}(\mathbb{C})$ are defined as

$$\sigma(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)} \quad \text{and} \quad \rho(g) := \limsup_{r\to\infty} \frac{\log^+ T(r,g)}{\log r}.$$

The *order of growth* of $f \in \mathcal{H}(\mathbb{D})$ is

$$\sigma_M(f) := \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) := \max_{|z|=r} |f(z)|$. It is well known that the inequalities $\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1$ are satisfied for all $f \in \mathcal{H}(\mathbb{D})$.

For $p > 0$ and $q > -1$, the *weighted Bergman space* A^p_q consists of those $h \in \mathcal{H}(\mathbb{D})$ for which

$$\|h\|_{A^p_q} := \left(\int_{\mathbb{D}} |h(z)|^p (1-|z|^2)^q \, dm(z)\right)^{\frac{1}{p}} < \infty.$$

Functions of maximal growth in $\bigcap_{\alpha<q<\infty} A^p_q$ are distinguished by denoting $h \in \mathbb{A}^p_\alpha$ if $\alpha = \inf\{q > -1 : h \in A^p_q\}$. Moreover, $h \in \mathcal{H}(\mathbb{D})$ belongs to H^∞_p, $0 \le p < \infty$, if

$$\|h\|_{H^\infty_p} := \sup_{z\in\mathbb{D}} |h(z)|(1-|z|^2)^p < \infty,$$

and $f \in \mathbb{H}^\infty_p$ if $p = \inf\{q \ge 0 : f \in H^\infty_q\}$.

The main results of this section are gathered to the following theorem.

Theorem 7. *Let $k \in \mathbb{N}$, $0 \le \alpha < \infty$ and $1 \le \beta < \infty$.*

(a) *Let $f \in \mathcal{R}_k(\mathbb{D})$. Then $\sigma(f) \le \alpha$ if and only if $S_k(f) \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$. In particular, if $\alpha > 0$, then $\sigma(f) = \alpha$ if and only if $S_k(f) \in \mathbb{A}_\alpha^{\frac{1}{k}}$.*

(b) *Let $f \in \mathcal{R}_1(\mathbb{D})$. Then $\sigma_M(f) \le \beta$ if and only if $S_k(f) \in \cap_{q>k(\beta+1)} H^\infty_q$. In particular, if $\beta > 1$, then $\sigma_M(f) = \beta$ if and only if $S_k(f) \in \mathbb{H}^\infty_{k(\beta+1)}$.*

(c) *Let $g \in \mathcal{R}_k(\mathbb{C})$. Then $\rho(g) \le \beta$ if and only if $S_k(g)$ is a polynomial with $\deg(S_k(g)) \le k(\beta-1)$. In particular, $\rho(g) = \beta$ if and only if $S_k(g)$ is a polynomial with $\deg(S_k(g)) = k(\beta-1)$.*

If $f \in \mathcal{R}_1(\mathbb{D}) \subset \mathcal{R}_k(\mathbb{D})$, then $\log f' \in \mathcal{H}(\mathbb{D})$. Corollary 8 is obtained from Theorem 7(a)(b) by applying the well-known inequalities

$$C_1^{-1}\|h\|_{A^p_\alpha} \leq \|h'\|_{A^p_{p+\alpha}} + |h(0)| \leq C_1\|h\|_{A^p_\alpha},$$
$$C_2^{-1}\|h\|_{H^\infty_\alpha} \leq \|h'\|_{H^\infty_{\alpha+1}} + |h(0)| \leq C_2\|h\|_{H^\infty_\alpha},$$

valid for all $h \in \mathcal{H}(\mathbb{D})$ and for some $C_1 > 0$, depending only on p and α, and $C_2 > 0$, depending only on α.

Corollary 8. *Let $f \in \mathcal{R}_1(\mathbb{D})$, $0 \leq \alpha < \infty$ and $1 \leq \beta < \infty$. Then $\sigma(f) \leq \alpha$ if and only if $\log f' \in \cap_{q>\alpha-1} A^1_q$. In particular, if $\alpha > 0$, then $\sigma(f) = \alpha$ if and only if $\log f' \in \mathbb{A}^1_{\alpha-1}$. Similarly, $\sigma_M(f) \leq \beta$ if and only if $\log f' \in \cap_{q>\beta} H^\infty_q$. In particular, if $\beta > 1$, then $\sigma_M(f) = \beta$ if and only if $\log f' \in \mathbb{H}^\infty_\beta$.*

Before analyzing Theorem 7(c), we give an example and shortly discuss conformal maps of $\mathbb{D}$.

Example 4. Let $f(z) = \exp(1/(1-z)^\gamma)$, where $\gamma > 1$. Then $f \in \mathcal{R}_1(\mathbb{D})$ and $\sigma(f) = \gamma - 1$. Moreover,

$$\begin{aligned}
\log f'(z) &= \frac{1}{(1-z)^\gamma} + \log \frac{\gamma}{(1-z)^{\gamma+1}}, \\
\frac{f''(z)}{f'(z)} &= \frac{\gamma}{(1-z)^{\gamma+1}} + \frac{\gamma+1}{1-z}, \\
S_f(z) &= -\frac{\gamma^2}{2(1-z)^{2\gamma+2}} - \frac{\gamma^2-1}{2(1-z)^2},
\end{aligned}$$

and hence $S_k(f) \in \mathbb{A}^{\frac{1}{k}}_{\gamma-1}$, $k = 1, 2$, and $\log f' \in \mathbb{A}^1_{\gamma-2}$ as Theorem 7 and Corollary 8 claim.

If f is a conformal map of $\mathbb{D}$ onto the inner domain of a Jordan curve $\mathcal{C}$, then geometric properties of $\mathcal{C}$ are related to analytic properties of $\log f'$ [18]. Moreover, several analytic properties of $\log f'$ (or f''/f') can been expressed in terms of the Schwarzian derivative S_f [1, 4, 16, 17, 18]. For example, $\log f'$ belongs to the classical Dirichlet space $\mathcal{D}$ (functions in $\mathcal{H}(\mathbb{D})$ with square integrable derivative) if and only if $S_f \in A^2_2$ [17]. If $S_f \in A^2_2$, then $|S_f(z)|$ is of the growth $o(1/(1-|z|^2)^2)$, yet all conformal maps f satisfy the well-known inequality $|S_f(z)| \leq 6/(1-|z|^2)^2$ for all $z \in \mathbb{D}$. It is obvious that the Schwarzian derivative of a function in $\mathcal{R}_1$ may have a much larger growth as the function f in Example 4 shows. It is also worth noticing that the methods of proof for conformal maps do not seem to yield Theorem 7(a)(b).

To see that the cases (a) and (c) of Theorem 7 are analogous, one only needs to notice that an entire function g is a polynomial with $\deg(g) \leq k(\beta-1)$ if and only if

$$\int_{\mathbb{C}\setminus\mathbb{D}} |g(z)|^{\frac{1}{k}} |z|^{-(\beta+1+\varepsilon)}\, dm(z) < \infty$$

for all $\varepsilon > 0$.

All values of β are not permitted in the case of equality $\rho(g) = \beta$ in Theorem 7(c). Namely, if $g \in \mathcal{R}_k(\mathbb{C})$ is not rational, then $\rho(g) \in \{1 + \frac{n}{k} : n = 0, 1, \ldots\}$. This is not a surprise, because the growth of g is determined via $g' = h^{-k}$ by a solution h of (1.3) with B entire. As $\rho(h) = \rho(g') = \rho(g) < \infty$, logarithmic derivative estimates (see Section 5.3 for similar reasonings) show that B must be a polynomial, and therefore the possible orders of solutions h are restricted to the values $1 + \frac{n}{k}$, $n = 0, 1, \ldots$, see [9] for a proof and a further discussion on the subject.

Oscillation of solutions of $h^{(k)} + B(z)\, h = 0$

Theorems 6 and 7 can be used to deduce known results on the oscillation of solutions of

$$h^{(k)} + B(z)\, h = 0. \tag{1.8}$$

To give the precise statement, definitions are needed. Let $\{z_n\}$ and $\{w_n\}$ be the zeros of $f \in \mathcal{H}(\mathbb{D})$ and $g \in \mathcal{H}(\mathbb{C})$, respectively. The *exponents of convergence* for the zeros of f and of g are defined as

$$\lambda(f) := \inf\left\{\alpha > 0 : \sum_{n=1}^{\infty}(1 - |z_n|)^{\alpha+1} < \infty\right\} \text{ and } \mu(g) := \inf\left\{\beta > 0 : \sum_{n=1}^{\infty}|w_n|^{-\beta} < \infty\right\}.$$

Theorem B. *Let $\alpha \geq 0$ and $\beta \geq 1$.*

(a) *Let $B \in \mathcal{H}(\mathbb{D})$. Then all solutions h of* (1.8) *satisfy $\lambda(h) \leq \alpha$ if and only if $B \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$.*

(b) *Let $B \in \mathcal{H}(\mathbb{C})$. Then all solutions h of* (1.8) *satisfy $\mu(h) \leq \beta$ if and only if B is a polynomial with* $\deg(B) \leq k(\beta - 1)$.

Theorem B is a special case of results in [10], where the oscillation of solutions of linear differential equation (1.7) is studied by using a representation of analytic coefficients $B_0, \ldots, B_{k-2}$ in terms of ratios of linearly independent solutions [13]. Therefore, to avoid unnecessary repetition, we merely sketch a proof of (a), and refer to [10] for a further discussion on the topic.

It is well known that $\lambda(h) \leq \sigma(h)$ for all $h \in \mathcal{H}(\mathbb{D})$. Therefore one implication in Theorem B(a) follows by the growth estimates for the solutions of (1.8), see Lemma D(a) below. Conversely, let $B \in \mathcal{H}(\mathbb{D})$ and assume all solutions h of (1.8) satisfy $\lambda(h) \leq \alpha \in [0, \infty)$. Let $\{h_1, \ldots, h_k\}$ be a solution base of (1.8). Then an application of the second main theorem of Nevanlinna shows that $\sigma(h_j/h_k) \leq \alpha$ for all $j = 1, \ldots, k-1$, see [10] for details. It follows that every primitive function f of the Wronskian determinant (1.6) satisfies $\sigma(f) \leq \alpha$. But now Theorem 6 states $f \in \mathcal{R}_k$ and $S_k(f) = kB$, from which Theorem 7 yields $B \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$ as claimed.

2. Proofs of Lemmas 2-5

2.1. Proof of Lemma 2

Assume first $f \in \mathcal{R}_2$, that is, $f \in \mathcal{M}$ and there exists $h \in \mathcal{H}$ such that $f' = 1/h^2$. Then f' is clearly non-vanishing. Moreover, if h does not vanish at a point α, then f' is analytic at α, and so is f. If h has a zero at a point α, then $h(\alpha) = h''(\alpha) = 0$ and $h'(\alpha) \neq 0$ by Definition 1. Therefore, in a neighborhood of α, the function h is of the form $h(z) = a_1(z-\alpha) + (z-\alpha)^3 H(z)$, where $a_1 \neq 0$ and H is analytic. Hence

$$f'(z) = \frac{1}{a_1^2(z-\alpha)^2}\left(\frac{1}{1+2(z-\alpha)^2 H(z) a_1^{-1} + (z-\alpha)^4 (H(z)a^{-1})^2}\right),$$

and it follows that

$$f(z) = -\frac{1}{a_1^2}(z-\alpha)^{-1} - \frac{2\,H(\alpha)}{a_1^3}(z-\alpha) - \frac{H'(\alpha)}{a_1^3}(z-\alpha)^2 - \dots .$$

Therefore f has simple poles at zeros of h and is analytic elsewhere. Thus $f \in \mathcal{R}$.

Conversely, assume $f \in \mathcal{R}$, and define $B := \frac{1}{2}S_2(f)$. According to Theorem A, equation (1.2) admits linearly independent solutions h_1 and h_2 such that $f = h_1/h_2$. Further, the Wronskian determinant $W(h_1,h_2) = h_1'h_2 - h_1h_2'$ is a non-zero constant, and hence

$$f' = \left(\frac{h_1}{h_2}\right)' = \frac{h_1'h_2 - h_1h_2'}{h_2^2} = \frac{W(h_1,h_2)}{h_2^2} = \frac{1}{h^2},$$

where $h := h_2/\sqrt{W(h_1,h_2)}$ is a well-defined analytic function. As h is a solution of (1.2), f satisfies conditions (i) and (ii) in Definition 1, see Example 1. Thus $f \in \mathcal{R}_2$.

2.2. Proof of Lemma 3

Let $f \in \mathcal{M}$ such that $f' = 1/h^n$ for some $h \in \mathcal{H}$ and $n \in \mathbb{N}$. We claim that

$$S_{k+1,n}(f) = -n\frac{h^{(k)}}{h} \tag{2.1}$$

for all $k \in \mathbb{N}$. As $S_k(f) = S_{k+1,k}(f)$, the assertion in Lemma 3 follows by taking $n = k$ in (2.1). To prove (2.1), note first that

$$f' = \frac{1}{h^n}, \quad f'' = -n\frac{h'}{h^{n+1}} \quad \text{and} \quad S_{2,n}(f) = \frac{f''}{f'} = -n\frac{h'}{h},$$

and so the identity (2.1) is valid for $k = 1$. Assume now (2.1) for $k = m \geq 1$. Then

$$\begin{aligned} S_{m+2,n}(f) &= \big(S_{m+1,n}(f)\big)' - \frac{1}{n}\frac{f''}{f'}S_{m+1,n}(f) \\ &= \left(-n\frac{h^{(m)}}{h}\right)' - \frac{1}{n}\left(-n\frac{h'}{h}\right)\left(-n\frac{h^{(m)}}{h}\right) \\ &= -n\,\frac{h^{(m+1)}h - h'h^{(m)}}{h^2} - n\,\frac{h'h^{(m)}}{h^2} = -n\,\frac{h^{(m+1)}}{h}, \end{aligned}$$

and therefore (2.1) is valid for $k = m+1$. The identity (2.1) follows by induction. Moreover, (1.4) shows that any constant multiple of $(f')^{-1/k}$ is a solution of (1.5).

2.3. Proof of Lemma 4

The following auxiliary result is needed.

Lemma 9. *If $f \in \mathcal{M}$ and $S_k(f) \in \mathcal{H}$, then all poles of f' are of order lk, where $l \in \mathbb{N}$.*

Proof. Let $f \in \mathcal{M}$ and $S_k(f) \in \mathcal{H}$. Assume on the contrary that f' has a pole of order p at a point α, and k is not a divisor of p. Then there exist $R > 0$ and a non-vanishing $H \in \mathcal{H}(D(\alpha, R))$ such that $f'(z) = H(z)\,(z-\alpha)^{-p}$ in the annulus $0 < |z-\alpha| < R$. For a fixed branch, define the non-vanishing $K_0 \in \mathcal{H}(D(\alpha, R))$ by $K_0(z) := (H(z))^{-1/k}$. Then, for a fixed branch, the function

$$h(z) := \frac{(z-\alpha)^{p/k}}{(H(z))^{1/k}} = (z-\alpha)^{p/k}\, K_0(z)$$

satisfies $h \in \mathcal{H}(\Omega)$ for $\Omega := \big\{z \in D(\alpha, R) \colon \operatorname{Re} z > \operatorname{Re} \alpha\big\}$. Therefore

$$f'(z) = \frac{H(z)}{(z-\alpha)^p} = \frac{1}{(z-\alpha)^p\, K_0^k(z)} = \frac{1}{h^k(z)}, \quad z \in \Omega,$$

and hence $S_k(f) = -kh^{(k)}/h$ in Ω by Lemma 3.

Differentiation gives

$$h'(z) = (z-\alpha)^{p/k-1} K_1(z),$$

where $K_1(z) := \frac{p}{k} K_0(z) + (z-\alpha)K_0'(z)$ satisfies $K_1 \in \mathcal{H}(D(\alpha, R))$ and $K_1(\alpha) \neq 0$. After $k-1$ more differentiations, we obtain

$$h^{(k)}(z) := (z-\alpha)^{p/k-k}\, K_k(z),$$

where $K_k \in \mathcal{H}(D(\alpha, R))$ and $K_k(\alpha) \neq 0$. Therefore

$$S_k(f)(z) = -k\frac{h^{(k)}(z)}{h(z)} = -\frac{k}{(z-\alpha)^k}\,\frac{K_k(z)}{K_0(z)}, \quad z \in \Omega,$$

where $K_k/K_0 \in \mathcal{H}(D(\alpha, R))$ and $K_k(\alpha)/K_0(\alpha) \neq 0$. It follows that $S_k(f)$ does not remain bounded as $z \to \alpha$ in Ω. This contradicts the assumption $S_k(f) \in \mathcal{H}$, and the assertion follows. □

We proceed to prove Lemma 4. If $f' = (P_{k-1})^{-k}$, where P_{k-1} is a polynomial with $\deg(P_{k-1}) \leq k-1$, then Lemma 3 yields $S_k(f) = -k\, P_{k-1}^{(k)}/P_{k-1} \equiv 0$.

Conversely, assume

$$S_k(f) = S_{k+1,k}(f) = \big(S_{k,k}(f)\big)' - \frac{1}{k}\,\frac{f''}{f'}\, S_{k,k}(f) \equiv 0.$$

By solving this equation we obtain $S_{k,k}(f) = P_0\,(f')^{\frac{1}{k}}$, where $P_0 \in \mathbb{C}$. Hence

$$S_{k,k}(f) = \big(S_{k-1,k}(f)\big)' - \frac{1}{k}\,\frac{f''}{f'}\, S_{k-1,k}(f) \equiv P_0\,(f')^{\frac{1}{k}},$$

which in turn gives $S_{k-1,k}(f) = (P_0 z + C)(f')^{\frac{1}{k}} =: P_1\,(f')^{\frac{1}{k}}$, where $C \in \mathbb{C}$ and P_1 is a polynomial with $\deg(P_1) \leq 1$. Continuing in this fashion we obtain

$$S_{2,k}(f) = \frac{f''}{f'} \equiv P_{k-2}\,(f')^{\frac{1}{k}},$$

where P_{k-2} is a polynomial with $\deg(P_{k-2}) \leq k-2$. Since f' is non-vanishing by the assumption, Lemma 9 implies that there exists $h \in \mathcal{H}$, $h \not\equiv 0$, such that $f' = h^{-k}$. It follows that

$$-k\,\frac{h'}{h} \equiv \frac{P_{k-2}}{h},$$

and hence $h' = -P_{k-2}/k$ outside of zeros of h. Because both h and P_{k-2} are analytic, we deduce $f' = h^{-k} = (P_{k-1})^{-k}$, where P_{k-1} is a polynomial with $\deg(P_{k-1}) \leq k-1$.

2.4. Proof of Lemma 5

Claims (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) follow from Lemma 3. Namely, if $f \in \mathcal{R}_k$, then f' is of the form $f' = 1/h^k$, where $h \in \mathcal{H}$ and at each l-fold zero of h, $h^{(k)}$ has at least l-fold zero. Identity (1.4) implies $S_k(f) \in \mathcal{H}$, and (1.5) shows that h is a solution of (1.3), where $B = \frac{1}{k} S_k(f) \in \mathcal{H}$.

Since (iii) $\Longrightarrow$ (i) is proved in Example 1, it remains to consider the claim (ii) $\Longrightarrow$ (iii). To see this, let $f \in \mathcal{M}$ such that f' is non-vanishing and $S_k(f) \in \mathcal{H}$. Then Lemma 9 shows that f' can be written in the form $f' = h^{-k}$, where $h \in \mathcal{H}$. But now h is a solution of (1.5) by Lemma 3, and thus $f \in \mathcal{R}_k$.

3. Proof of the assertion in Example 3

If f' is analytic at α, then, for a fixed branch, $h = (f')^{-1/k}$ is analytic and non-vanishing in a neighborhood of α. Lemma 3 implies that $S_k(f) = -kh^{(k)}/h$ is analytic at α. If f' has a pole at α, then the Laurent series of f' in a neighborhood of α is of the form

$$f'(z) = \frac{c_{-lk}}{(z-\alpha)^{lk}} + \sum_{j=-lk+k}^{\infty} c_j (z-\alpha)^j =: \frac{c_{-lk}}{(z-\alpha)^{lk}} + H(z),$$

where $c_{-lk} \neq 0$ and $l \in \{1, 2, \dots, k-1\}$. Therefore $f' = 1/h^k$, where

$$h(z) = \frac{(z-\alpha)^l}{\left(c_{-lk} + (z-\alpha)^{lk} H(z)\right)^{1/k}} \tag{3.1}$$

is analytic at α. We may write

$$\left(c_{-lk} + (z-\alpha)^{lk} H(z)\right)^{1/k} = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots =: a_0 + A(z),$$

and hence

$$c_{-lk} + (z-\alpha)^{lk} H(z) = a_0^k + k a_0^{k-1} A(z) + \cdots + k a_0 A^{k-1}(z) + A^k(z).$$

From this equality, it follows that $a_0 = c_{-lk}^{\frac{1}{k}}$ and

$$A(z) = a_k(z-\alpha)^k + a_{k+1}(z-\alpha)^{k+1} + \cdots .$$

Now

$$\frac{1}{(c_{-lk} + (z-\alpha)^{lk}H(z))^{1/k}} = \frac{1}{a_0 + A(z)} = \sum_{j=0}^{\infty} b_j(z-\alpha)^j,$$

where $b_0 = a_0^{-1}$ and $b_j = -b_0 \sum_{n=1}^{j} a_n b_{j-n}$. Therefore $b_j = 0$ for all $j = 1, \ldots, k-1$, and hence (3.1) implies

$$h(z) = b_0(z-\alpha)^l + b_k(z-\alpha)^{l+k} + b_{k+1}(z-\alpha)^{l+k+1} + \cdots .$$

As $l \leq k-1$ by the assumption, differentiation gives

$$h^{(k)}(z) = d_l(z-\alpha)^l + d_{l-1}(z-\alpha)^{l+1} + \cdots ,$$

from which (3.1) yields

$$\frac{h^{(k)}(z)}{h(z)} = (a_0 + A(z))\left(d_l + d_{l+1}(z-\alpha) + d_{l+2}(z-\alpha)^2 + \cdots\right).$$

Therefore $h^{(k)}/h$ has a removable singularity at α, and so does $S_k(f)$ by Lemma 3. Hence $S_k(f) \in \mathcal{H}$, and Lemma 5 yields $f \in \mathcal{R}_k$.

4. Proof of Theorem 6

Let first $\{h_1, \ldots, h_k\}$ be a solution base of (1.3), where $B \in \mathcal{H}$. By [15, Proposition 1.4.3(e)] the Wronskian determinant (1.6) is of the form

$$f' = W\left(\left(\frac{h_1}{h_k}\right)', \left(\frac{h_2}{h_k}\right)', \ldots, \left(\frac{h_{k-1}}{h_k}\right)'\right) = \frac{1}{h_k^k}\, W(h_1, h_2, \ldots, h_k) = \frac{C}{h_k^k} = \left(\frac{h_k}{C^{1/k}}\right)^{-k},$$

where $C \in \mathbb{C} \setminus \{0\}$. But now $f \in \mathcal{R}_k$ by Lemma 5, and

$$S_k(f) = -k\,\frac{h_k^k}{h_k} = k\,B$$

by Lemma 3 since h_k is a solution of (1.3).

Conversely, let $f \in \mathcal{R}_k$ and $B = \frac{1}{k} S_k(f)$. Then $B \in \mathcal{H}$ by Lemma 5, and $h_k := (f')^{-1/k}$ is a solution of (1.3) by Lemma 3. Let $\{h_1, \ldots, h_k\}$ be a solution base of (1.3) with the normalization $W(h_1, h_2, \ldots, h_k) = 1$. According to [15, Proposition 1.4.3(e)],

$$W\left(\left(\frac{h_1}{h_k}\right)', \left(\frac{h_2}{h_k}\right)', \ldots, \left(\frac{h_{k-1}}{h_k}\right)'\right) = \frac{1}{h_k^k}\, W(h_1, h_2, \ldots, h_k) = \frac{1}{h_k^k} = \frac{1}{((f')^{-1/k})^k} = f',$$

which completes the proof.

5. Proof of Theorem 7

We begin with the following lemma which contains different logarithmic derivative estimates needed when proving the different cases of Theorem 7. For proofs of these estimates, see [6, 7, 8]. Recall that the *upper density* of a measurable set $E \subset [0,1)$ is defined as

$$\overline{D}(E) = \limsup_{r\to 1^-} \frac{m(E\cap[r,1))}{1-r},$$

where $m(F)$ denotes the Lebesgue measure of the set F.

Lemma C. *Let k and j be integers satisfying $k > j \geq 0$. Let $f \in \mathcal{M}(\mathbb{D})$ such that $\sigma(f) < \infty$ and $f^{(j)} \not\equiv 0$, and let $g \in \mathcal{M}(\mathbb{C})$ such that $\rho(g) < \infty$ and $g^{(j)} \not\equiv 0$.*

(a) *Then*

$$\int_{\mathbb{D}} \left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right|^{\frac{1}{k-j}} (1-|z|^2)^{\sigma(f)+\varepsilon}\, dm(z) < \infty$$

for all $\varepsilon > 0$.

(b) *For given $\varepsilon > 0$ and $0 < \delta < 1$, there exists a set $E \subset [0,1)$ satisfying $\overline{D}(E) < \delta$ such that*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq \left(\frac{1}{1-|z|} \right)^{(\max\{\sigma_M(f),1\}+1)(k-j)+\varepsilon}$$

for all $z \in \mathbb{D}$ with $|z| \notin E$.

(c) *For a given $\varepsilon > 0$ there exists a set $E \subset (1,\infty)$ satisfying $\int_E \frac{dr}{r} < \infty$ such that*

$$\left| \frac{g^{(k)}(z)}{g^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(g)-1+\varepsilon)}$$

for all $z \in \mathbb{C}$ with $|z| \notin E \cup [0,1]$.

Another auxiliary result needed concerns finite order solutions of the linear differential equation

$$h^{(k)} + B_{k-1}(z)h^{(k-1)} + \cdots + B_1(z)h' + B_0(z)h = 0 \qquad (5.1)$$

with analytic coefficients $B_0, \ldots, B_{k-1}$. The following lemma follows at once by [11, Theorem 4.1]. For earlier results and further studies on the topic, see [6, 7, 9, 12, 14, 15, 20] and the references therein.

Lemma D. *Let $0 \leq \alpha < \infty$ and $1 \leq \beta < \infty$.*

(a) *If $B_j \in \bigcap_{q>\alpha} A_q^{\frac{1}{k-j}}$ for all $j = 0, \ldots, k-1$, then all solutions f of (5.1) satisfy $\sigma(f) \leq \alpha$.*

(b) *If $B_j \in \bigcap_{q>(k-j)(\beta+1)} H_q^\infty$ for all $j = 0, \ldots, k-1$, then all solutions f of (5.1) satisfy $\sigma_M(f) \leq \beta$.*

(c) *If B_j is a polynomial with $\deg(B_j) \leq (k-j)(\beta-1)$ for all $j = 0, \ldots, k-1$, then all solutions f of (5.1) satisfy $\rho(f) \leq \beta$.*

5.1. Proof of Theorem 7(a)

Let first $f \in \mathcal{R}_k(\mathbb{D})$ such that $\sigma(f) \leq \alpha \in [0,\infty)$. Then f' can be written in the form $f' = 1/h^k$, where $h \in \mathcal{H}(\mathbb{D})$ admits the properties (i) and (ii) of Definition 1. Moreover, $h = (f')^{-1/k}$ satisfies $\sigma(h) \leq \alpha$. Lemma 3 and Lemma C(a) now yield

$$\int_{\mathbb{D}} |S_k(f)(z)|^{\frac{1}{k}} (1-|z|^2)^{\alpha+\varepsilon}\, dm(z) = k^{\frac{1}{k}} \int_{\mathbb{D}} \left| \frac{h^{(k)}(z)}{h(z)} \right|^{\frac{1}{k}} (1-|z|^2)^{\alpha+\varepsilon}\, dm(z) < \infty$$

for all $\varepsilon > 0$. As $S_k(f) \in \mathcal{H}(\mathbb{D})$ by Lemma 5, $S_k(f) \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$.

Conversely, if $S_k(f) \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$, then all solutions h of

$$h^{(k)} + \frac{1}{k} S_k(f)(z)\, h = 0 \tag{5.2}$$

are analytic and satisfy $\sigma(h) \leq \alpha$ by Lemma D(a). As $h = (f')^{-1/k}$ is one of the solutions by Lemma 3, this yields $\sigma(f) = \sigma((f')^{-1/k}) \leq \alpha$.

Let now $\alpha > 0$, and let $f \in \mathcal{R}_k(\mathbb{D})$ such that $\sigma(f) = \alpha$. Then $S_k(f) \in \cap_{q>\alpha} A_q^{\frac{1}{k}}$ by the proof above. If $S_k(f) \in A_{\alpha-\varepsilon}^{\frac{1}{k}}$ for some $\varepsilon > 0$, then all solutions h of (5.2) are analytic and satisfy $\sigma(h) \leq \alpha - \varepsilon$ by Lemma D(a). Since $h = (f')^{-1/k}$ is one of the solutions by Lemma 3, this yields $\alpha = \sigma(f) = \sigma((f')^{-1/k}) \leq \alpha - \varepsilon$. This is clearly a contradiction, and thus $S_k(f) \in \mathbb{A}_\alpha^{\frac{1}{k}}$.

Conversely, if $S_k(f) \in \mathbb{A}_\alpha^{\frac{1}{k}}$, then the proof above shows that $\sigma(f) \leq \alpha$. Moreover, if $\sigma(f) < \alpha$, then $S_k(f) \in A_{\alpha-\varepsilon}^{\frac{1}{k}}$ for some $\varepsilon > 0$ by Lemma C(a). This clearly contradicts the assumption $S_k(f) \in \mathbb{A}_\alpha^{\frac{1}{k}}$, and thus $\sigma(f) = \alpha$.

5.2. Proof of Theorem 7(b)

We will need the following auxiliary result [7, Lemma 4.1] to deal with the exceptional set which appears in Lemma C(b).

Lemma E. *Let $B \in \mathbb{H}_\alpha^\infty$ for some $\alpha \in (0,\infty)$. For given $\varepsilon > 0$ and $\delta \in (0,1)$, there exists a set $F \subset [0,1)$ with $\overline{D}(F) \geq \delta$ such that*

$$\liminf_{\substack{r\to 1^- \\ r\in F}} \frac{\log^+ M(r,B)}{-\log(1-r)} \geq \alpha - \varepsilon.$$

To prove Theorem 7(b), let first $f \in \mathcal{R}_1(\mathbb{D})$ such that $\sigma_M(f) \leq \beta \in [1,\infty)$, and let $\varepsilon > 0$. Then for given $k \in \mathbb{N}$, f' can be written in the form $f' = 1/h^k$, where $h \in \mathcal{H}(\mathbb{D})$ is non-vanishing, and $\sigma_M(f') \leq \beta$. It follows that $\left|\mathrm{Re}\big(\log^+ f'(re^{i\theta})\big)\right| = O\left((1-r)^{-\beta-\varepsilon}\right)$. Since $\log f' \in \mathcal{H}(\mathbb{D})$, inequality (1.18) in [5] now yields

$$\log M\left(r, 1/f'\right) \leq M\left(r, \log 1/f'\right) = M\left(r, \log f'\right) = O\left(\frac{1}{(1-r)^{\beta+\varepsilon}}\right).$$

Hence $\sigma_M(1/f') \leq \beta$, and

$$\sigma_M(h) = \sigma_M((f')^{-1/k}) = \sigma_M(1/f') \leq \beta.$$

By Lemma 3 and Lemma C(b), for given $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a set $E \subset [0,1)$ satisfying $\overline{D}(E) < \delta$ such that

$$|S_k(f)(z)| = k\left|\frac{h^{(k)}(z)}{h(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{k(\max\{\sigma_M(h),1\}+1)+\varepsilon} \leq \left(\frac{1}{1-|z|}\right)^{k(\beta+1)+\varepsilon} \tag{5.3}$$

for all $z \in \mathbb{D}$ with $|z| \notin E$. Moreover, $S_k(f) \in \mathcal{H}(\mathbb{D})$ by Lemma 5. Assume on the contrary that $S_k(f) \in \mathbb{H}^\infty_\alpha$ for some $\alpha > k(\beta+1)$. Fix $\varepsilon > 0$ such that $2\varepsilon < \alpha - k(\beta+1)$. By Lemma E there exists a set $F \subset [0,1)$ satisfying $\overline{D}(F) \geq 2\delta$ such that

$$\liminf_{\substack{r\to 1^- \\ r\in F}} \frac{\log^+ M(r, S_k(f))}{-\log(1-r)} \geq \alpha - \varepsilon. \tag{5.4}$$

Combining (5.3) and (5.4) we find $\{r_n\} \subset F \setminus E$ with $r_n \to 1^-$, as $n \to \infty$, such that

$$\left(\frac{1}{1-r_n}\right)^{\alpha-\varepsilon} \leq M(r_n, S_k(f)) \leq \left(\frac{1}{1-r_n}\right)^{k(\beta+1)+\varepsilon}, \quad n \in \mathbb{N}.$$

Since $\alpha - \varepsilon > k(\beta+1)+\varepsilon$, this yields a contradiction, and therefore $S_k(f) \in \mathbb{H}^\infty_\alpha$ for some $\alpha \leq k(\beta+1)$. Thus $S_k(f) \in \cap_{q>k(\beta+1)} H^\infty_q$.

Conversely, if $S_k(f) \in \cap_{q>k(\beta+1)} H^\infty_q$, then all solutions h of (5.2) are analytic and satisfy $\sigma_M(h) \leq \beta$ by Lemma D(b). As $h = (f')^{-1/k}$ is one of the solutions by Lemma 3, this yields

$$\sigma_M(1/f') = \sigma_M((f')^{-1/k}) = \sigma_M(h) \leq \beta.$$

By an argument similar to the one given in the beginning of the proof of Theorem 7(b), we see that $\sigma_M(f) = \sigma_M(f') \leq \beta$. The assertion on the case of equality can be proved by following the corresponding reasoning in the proof of Theorem 7(a).

5.3. Proof of Theorem 7(c)

We will need one more auxiliary result [2, 8].

Lemma F. *Let φ and ψ be monotone increasing functions on $[0,\infty)$ such that $\varphi(r) \leq \psi(r)$ for all $r \notin E \cup [0,1]$, where $E \subset (1,\infty)$ satisfies $\int_E \frac{dr}{r} < \infty$. Then, for any $\gamma > 1$ there exists $r_\gamma > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r \in [r_\gamma, \infty)$.*

To prove Theorem 7(c), let first $g \in \mathcal{R}_k(\mathbb{C})$ such that $\rho(g) \leq \beta \in [1,\infty)$. Then g' can be written in the form $g' = 1/h^k$, where $h \in \mathcal{H}(\mathbb{C})$ admits the properties (i) and (ii) of Definition 1. Moreover, $h = (g')^{-1/k}$ satisfies $\rho(h) \leq \beta$. Lemma 3 and Lemma C(c) now yield

$$|S_k(g)(z)| = k\left|\frac{h^{(k)}(z)}{h(z)}\right| \leq |z|^{k(\beta-1+\varepsilon)},$$

provided $|z| \notin E \cup [0,1]$, where $\int_E \frac{dr}{r} < \infty$. Lemma F now gives $M(r, S_k(g)) \leq r^{k(\beta-1+2\varepsilon)}$ for all r sufficiently large. Since $S_k(g) \in \mathcal{H}(\mathbb{C})$ by Lemma 5, this means that $S_k(g)$ is a polynomial with $\deg(S_k(g)) \leq k(\beta - 1)$.

Conversely, if $S_k(g)$ is a polynomial with $\deg(S_k(g)) \leq k(\beta - 1)$, then all solutions h of

$$h^{(k)} + \frac{1}{k} S_k(g)(z) h = 0$$

are entire and satisfy $\rho(h) \leq \beta$ by Lemma D(c). As $h = (g')^{-1/k}$ is one of the solutions by Lemma 3, this yields $\rho(g) = \rho((g')^{-1/k}) \leq \beta$.

The assertion on the case of equality can be proved by following the corresponding reasoning in the proof of Theorem 7(a).

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